Consider paraparticle state

$$|\Psi'\rangle = 1/\sqrt{2}\,\big(|a^r\rangle \otimes |a^s\rangle \otimes |a^n\rangle - |a^s\rangle \otimes |a^r\rangle \otimes |a^n\rangle\big)$$

$$\mathrm{Prob}^{|\Psi'\rangle}(Q_1 = q^\alpha,\ Q_2 = q^\beta,\ Q_3 = q^\gamma)$$

$$= \big|\langle q^\alpha| \otimes \langle q^\beta| \otimes \langle q^\gamma| \,(|\Psi'\rangle)\big|^2$$

$$= \frac{1}{2}\Big[|\langle q^\alpha|a^r\rangle|^2 \cdot |\langle q^\beta|a^s\rangle|^2 \cdot |\langle q^\gamma|a^r\rangle|^2$$
$$+ |\langle q^\alpha|a^s\rangle|^2 \cdot |\langle q^\beta|a^r\rangle|^2 \cdot |\langle q^\gamma|a^n\rangle|^2$$
$$- 2\,\mathrm{Re}\,\langle a^r|q^\alpha\rangle\langle q^\alpha|a^s\rangle \cdot \langle a^s|q^\beta\rangle\langle q^\beta|a^r\rangle \cdot \langle a^n|q^\gamma\rangle\langle q^\gamma|a^n\rangle\Big]$$

$$\therefore \mathrm{Prob}^{|\Psi'\rangle}(Q_1 = q^\alpha \;\&\; Q_2 = q^\beta) = \frac{1}{2}\Big[|\langle q^\alpha|a^r\rangle|^2 \cdot |\langle q^\beta|a^s\rangle|^2$$
$$+ |\langle q^\alpha|a^s\rangle|^2 \cdot |\langle q^\beta|a^r\rangle|^2$$
$$- 2\,\mathrm{Re}\,\langle a^r|q^\alpha\rangle\langle q^\alpha|a^s\rangle \cdot \langle a^s|q^\beta\rangle \cdot \langle q^\beta|a^r\rangle\Big]$$

$$= \mathrm{Prob}^{|\Psi'\rangle}(Q_1 = q^\beta \;\&\; Q_2 = q^\alpha)$$

$$\therefore \mathrm{Prob}(Q_1 = q^\alpha \mid Q_2 = q^\beta) = \mathrm{Prob}(Q_2 = q^\alpha \mid Q_1 = q^\beta)$$

$$= \Big[|\langle q^\alpha|a^r\rangle|^2 \cdot |\langle q^\beta|a^s\rangle|^2$$
$$+ |\langle q^\alpha|a^s\rangle|^2 \cdot |\langle q^\beta|a^r\rangle|^2$$
$$- 2\,\mathrm{Re}\,\langle a^r|q^\alpha\rangle\langle q^\alpha|a^s\rangle\langle a^s|q^\beta\rangle\langle q^\beta|a^r\rangle\Big]$$
$$\Big/ \Big[|\langle q^\beta|a^r\rangle|^2 + |\langle q^\beta|a^s\rangle|^2\Big]$$

Also $|\Psi'\rangle$

$\mathrm{Prob}(Q_1 = q^\beta , Q_3 = q^\gamma)$

$$= \frac{1}{2}\Big[ |\langle q^\beta | a^s\rangle|^2 \cdot |\langle q^\gamma | a^r\rangle|^2 + |\langle q^\beta | a^r\rangle|^2 |\langle q^\gamma | a^r\rangle|^2 \Big]$$

$$= \frac{1}{2} |\langle q^\gamma | a^r\rangle|^2 \Big[ |\langle q^\beta | a^s\rangle|^2 + |\langle q^\beta | a^r\rangle|^2 \Big]$$

$$= \mathrm{Prob}^{|\Psi'\rangle}(Q_1 = q^\beta , Q_3 = q^\gamma)$$

$$\therefore \mathrm{Prob}^{|\Psi'\rangle}(Q_1 = q^\beta \,|\, Q_3 = q^\gamma)$$

$$= \mathrm{Prob}^{|\Psi'\rangle}(Q_1 = q^\beta \,|\, Q_3 = q^\gamma)$$

$$= \frac{1}{2}\Big[ |\langle q^\beta | a^s\rangle|^2 + |\langle q^\beta | a^r\rangle|^2 \Big]$$

which is independent of of $q^\gamma$ and is also equal to

$\mathrm{Prob}^{|\Psi\rangle}(Q_1 = q^\beta \,|\, Q_3 = q^\gamma)$

Finally $\mathrm{Prob}^{\Psi'}(Q_1 = q^\alpha \,|\, Q_2 = q^\beta , Q_3 = q^\gamma)$

$$= \Big[ |\langle q^\alpha | a^r\rangle|^2 \cdot |\langle q^\beta | a^s\rangle|^2 |\langle q^\gamma | a^r\rangle|^2 + |\langle q^\alpha | a^s\rangle|^2 \cdot |\langle q^\beta | a^r\rangle|^2 |\langle q^\gamma | a^r\rangle|^2 - 2\,\mathrm{Re}\, \langle a^r | q^\alpha\rangle\langle q^\alpha | a^s\rangle\langle a^s | q^\beta\rangle\langle q^\beta | a^r\rangle \langle a^r | q^\gamma\rangle\langle q^\gamma | a^r\rangle \Big]$$

$$\Big/ |\langle q^\gamma | a^r\rangle|^2 \Big[ |\langle q^\beta | a^s\rangle|^2 + |\langle q^\beta | a^r\rangle|^2 \Big]$$

$$= \mathrm{Prob}^{|\Psi'\rangle}(Q_2 = q^\alpha \,|\, Q_1 = q^\beta , Q_3 = q^\gamma)$$

we have · with new Double Slit state $|\Psi'\rangle$.

$$\text{Prob}\left(Q_1 = q^\alpha \ \& \ Q_3 = q^\gamma\right)$$

$$= \frac{1}{2}\Big[\,|\langle q^\alpha|a^r\rangle|^2 \cdot |\langle q^\gamma|a^s\rangle|^2$$

$$+ \,|\langle q^\alpha|a^s\rangle|^2 \cdot |\langle q^\gamma|a^r\rangle|^2$$

$$- 2\,\text{Re}\,\langle a^r|q^\alpha\rangle\langle q^\alpha|a^s\rangle$$

$$\cdot \langle a^s|q^\gamma\rangle\langle q^\gamma|a^r\rangle\Big]$$

$$\text{Prob}\left(Q_1 = q^\alpha \ \& \ Q_2 = q^\beta\right)$$

$$= \frac{1}{2}\,|\langle q^\beta|a^r\rangle|^2\Big[\,|\langle q^\alpha|a^r\rangle|^2$$

$$+ \,|\langle q^\alpha|a^s\rangle|^2\Big]$$

One-particle and 2-particle distributions for Boson & Fermion states

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Consider state $|\Psi(1,2)\rangle = \frac{1}{\sqrt{2}}\left(|a_1^1\rangle|a_2^2\rangle \pm |a_1^2\rangle|a_2^1\rangle\right)$

Consider measurement magnitudes $Q_1$ and $Q_2$

Then $\text{Prob}^{\Psi}(Q_1 = q) = \frac{1}{2}\left(|a^1(q)|^2 + |a^2(q)|^2\right)$

where $a^1(q) =_{Df} \langle q|a^1\rangle$

$a^2(q) =_{Df} \langle q|a^2\rangle$

Similarly

$\text{Prob}^{\Psi}(Q_2 = q) = \frac{1}{2}\left(|a^1(q)|^2 + |a^2(q)|^2\right)$

Joint Probability

$\text{Prob}^{\Psi}(Q_1 = q' \;\&\; Q_2 = q'')$

$$= \frac{1}{2}\left|a^1(q')a^2(q'') \pm a^2(q')a^1(q'')\right|^2$$

$\left(= \text{Prob}^{\Psi}(Q_1 = q'' \;\&\; Q_2 = q')\right)$

$\therefore \text{Prob}^{\Psi}(Q_1 = q' / Q_2 = q'')$

~~But when~~

$$= \frac{|a^1(q')a^2(q'') \pm a^2(q')a^1(q'')|^2}{|a^1(q'')|^2 + |a^2(q'')|^2}$$

But notice that

$\text{Prob}^{\Psi}(Q_2 = q' | Q_2 = q'') = \underline{\delta_{q',q''}}$

Note that for the Fermion case:

$\mathrm{Prob}^{\Psi}(Q_1 = q' / Q_2 = q'')$

vanishes for $q' = q''$

there is the sense in which one might want to claim that they cannot have the same Q-value and hence the Leibniz law is contradicted – cf the Shalini discussion

Consider a state $|\Psi\rangle$ of two particles labelled 1 and 2 of the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|a^r\rangle|a^s\rangle \pm |a^s\rangle|a^r\rangle\right)$$

where $a^t$ is the $t^{th}$ eigenvalue of some maximal observable A, ~~with which we~~ ~~again~~ with discrete spectrum. We use the convention that in a tensor product the ~~first~~ left-hand member refers to particle 1 and the right-hand member to particle 2.

Consider now another maximal observable Q with eigenvalues $q^u$ (we again assume a discrete spectrum for simplicity), which we write as $Q_1$ when relating to particle 1 or $Q_2$ when relating to particle 2.

Then we have ~~the result~~ for the joint ~~$\mathrm{Prob}^{|\Psi\rangle}(Q$~~ probability that in the state $|\Psi\rangle$, $Q_1$ will yield the result $q^\alpha$ on measurement and $Q_2$ the result $q^\beta$

$$\mathrm{Prob}^{|\Psi\rangle}(Q_1 = q^\alpha \ \& \ Q_2 = q^\beta)$$

$$= \left|\langle q^\alpha|\langle q^\beta|\left(|\Psi\rangle\right)\right|^2$$

$$= \tfrac{1}{2}\left|\langle q^\alpha|a^r\rangle\langle q^\beta|a^s\rangle \pm \langle q^\alpha|a^s\rangle\langle q^\beta|a^r\rangle\right|^2$$

$$= \tfrac{1}{2}\langle a^r|q^\alpha\rangle\langle q^\alpha|a^r\rangle\langle a^s|q^\beta\rangle\langle q^\beta|a^s\rangle$$

$$+ \tfrac{1}{2}\langle a^s|q^\alpha\rangle\langle q^\alpha|a^s\rangle\langle a^r|q^\beta\rangle\langle q^\beta|a^r\rangle$$

~~$\pm 2\,\mathrm{Re}\,\langle a^r|q^\alpha\rangle\langle a^s|q^\beta\rangle\langle q$~~

$$\pm \cancel{2}\,\mathrm{Re}\,\langle a^r|q^\alpha\rangle\langle q^\alpha|a^s\rangle\cdot\langle a^s|q^\beta\rangle\langle q^\beta|a^r\rangle$$

From this basic result we can easily compute the marginal probabilities

$$\mathrm{Prob}^{|\Psi\rangle}(Q_1 = q^{\mu}) = \mathrm{Prob}^{|\Psi\rangle}(Q_2 = q^{\mu})$$

$$\cancel{= \frac{1}{2} \langle a^r | q^{\alpha} \rangle \langle q^{\alpha}}$$

$$= \frac{1}{2} \langle a^r | q^{\mu} \rangle \langle q^{\mu} | a^r \rangle$$

$$+ \frac{1}{2} \langle a^s | q^{\mu} \rangle \langle q^{\mu} | a^s \rangle$$

$$= \frac{1}{2} |\langle q^{\mu} | a^r \rangle|^2 + \frac{1}{2} |\langle q^{\mu} | a^s \rangle|^2.$$

We can also compute the conditional probabilities

$$\mathrm{Prob}^{|\Psi\rangle}(Q_1 = q^{\alpha} \mid Q_2 = q^{\beta})$$

$$= \frac{\mathrm{Prob}^{\Psi}(Q_1 = q^{\alpha} \;\&\; Q_2 = q^{\beta})}{\mathrm{Prob}^{\Psi}(Q_2 = q^{\beta})}$$

$$\cancel{= \frac{1}{2} |\langle q^{\alpha} | a^r \rangle|^2 + \frac{1}{2} |\langle}$$

$$= \Big[ \cancel{\tfrac{1}{2}} |\langle q^{\alpha} | a^r \rangle|^2 \cdot |\langle q^{\beta} | a^s \rangle|^2$$

$$+ \cancel{\tfrac{1}{2}} |\langle q^{\alpha} | a^s \rangle|^2 \cdot |\langle q^{\beta} | a^r \rangle|^2$$

$$\pm 2 \,\mathrm{Re}\, \langle a^r | q^{\alpha} \rangle \langle q^{\alpha} | a^s \rangle \langle a^s | q^{\beta} \rangle \langle q^{\beta} | a^r \rangle \Big]$$

$$\Big/ \; |\langle q^{\beta} | a^r \rangle|^2 + |\langle q^{\beta} | a^s \rangle|^2$$

It is easily checked that ~~this~~ formula

this equals $\mathrm{Prob}^{|\Psi\rangle}(Q_2 = q^\alpha \mid Q_1 = q^\beta)$
so we have the important relation

$$\mathrm{Prob}^{|\Psi\rangle}(Q_1 = q^\alpha \mid Q_2 = q^\beta)$$
$$= \mathrm{Prob}^{|\Psi\rangle}(Q_2 = q^\alpha \mid Q_1 = q^\beta)$$